

The boundary field theory induced by the Chern-Simons theory

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Abstract. The Chern-Simons theory defined on a 3-dimensional manifold with boundary is written as a two-dimensional field theory defined only on the boundary of the three-manifold. The resulting theory is, essentially, the pull-back to the boundary of a symplectic structure defined on the space of auxiliary fields in terms of which the connection one-form of the Chern-Simons theory is expressed when solving the condition of vanishing curvature. The counting of the physical degrees of freedom living in the boundary associated to the model is performed using Dirac's canonical analysis for the particular case of the gauge group $SU(2)$. The result is that the specific model has one physical local degree of freedom. Moreover, the role of the boundary conditions on the original Chern-Simons theory is displayed and clarified in an example, which shows how the gauge content as well as the structure of the constraints of the induced boundary theory is affected.

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1. Introduction

As is well-known, Chern-Simons theory defined on a three-dimensional manifold \mathcal{M}^3 *without* boundary has no local degrees of freedom [1, 2, 3]. The only physical degrees of freedom of the theory are global and associated with the topological properties of the three-manifold. Nevertheless, when the three-manifold \mathcal{M}^3 has a boundary $\partial\mathcal{M}^3$ then the theory has local physical degrees of freedom living at $\partial\mathcal{M}^3$. This fact is also very well-known in the literature [3, 4, 5]. In spite of this, the origin and the meaning of the physical degrees of freedom has not been completely elucidated, as far as we know.

The purpose of this paper is to make a contribution to this end. Our analysis does not hold for arbitrary three-dimensional manifolds with boundary. However, it has the advantage of displaying without ambiguities the field theory living at the boundary when the current analysis holds. More precisely, the goal of this paper is to construct a *covariant* action principle living at the boundary $\partial\mathcal{M}^3$ of the three-manifold \mathcal{M}^3 by solving the equations of motion $F = 0$ and expressing the connection field in terms of auxiliary fields. The resulting action principle for the field theory living at $\partial\mathcal{M}^3$ will depend functionally on these auxiliary fields. Once we have at our hand the covariant action at $\partial\mathcal{M}^3$, we count the number of local degrees of freedom of the theory by employing Dirac's canonical analysis.

It is worth noticing that the usual approaches for studying the local degrees of freedom at the boundary involve the following features [4, 5, 6, 7]: (1) they explicitly break covariance of the connection one-form, (2) related with item (1) is the fact that solutions to Dirac constraints are used to build the boundary action, (3) the handle of boundary terms and boundary conditions is not systematic. On the contrary, our approach leads to the construction of a covariant action principle. In our opinion, the knowledge of the Lagrangian action principle is relevant for various reasons, among them: it can be used to couple other interactions to it or to express already known actions (for instance, Palatini action in three-dimensional general relativity) in terms of it. Moreover, in our method, boundary terms and boundary conditions can be systematically implement through Dirac's canonical analysis as primary constraints [see, however, [8, 9, 10]].

2. The boundary field theory induced by an $SU(2)$ Chern-Simons theory

The starting point is the action principle

$$S[A^i] = \frac{\kappa}{4\pi} \int_{\mathcal{M}^3} \left[A^i \wedge F^j - \frac{1}{3!} f^i{}_{kl} A^k \wedge A^l \wedge A^j \right] k_{ij}, \quad (1)$$

where $F^i = dA^i + \frac{1}{2} f^i{}_{jk} A^j \wedge A^k$ is the curvature of the connection one-form $A = A^i J_i$, J_i are the generators of the Lie algebra and satisfy $[J_i, J_j] = f^k{}_{ij} J_k$ with $f^k{}_{ij}$ the structures constants of the Lie algebra, $k_{ij} = -\frac{1}{2} f^k{}_{il} f^l{}_{jk}$ is the Killing-Cartan metric and the constant κ is the so-called level of the theory. It is assumed that \mathcal{M}^3 has a boundary, $\partial\mathcal{M}^3 \neq \emptyset$.

The variation of the action principle (1) respect to the gauge connection A gives rise to the equations of motion $F^i = 0$. This tells us that the space of solutions is the space of flat connections. Consequently, the theory has no local physical degrees of freedom in the bulk and any possible local physical degree of freedom is contained at the boundary [2, 3, 4].

For the sake of simplicity, let us restrict the analysis to the gauge group $SU(2)$. In particular, the Lie algebra is spanned by the traceless skew-Hermitian 2×2 matrices $J_i = -\frac{i}{2}\sigma_i$, $i = 1, 2, 3$, which satisfy $[J_i, J_j] = \varepsilon^k{}_{ij}J_k$ where $\varepsilon^k{}_{ij}$ are the structure constants ($\varepsilon^1{}_{23} = -\varepsilon^2{}_{13} = \varepsilon^3{}_{12} = 1$) and σ_i are the Pauli matrices; $su(2)$ indices i, j, k, \dots are raised and lowered with the Killing-Cartan metric $\delta_{ij} = -\frac{1}{2}\varepsilon^k{}_{il}\varepsilon^l{}_{jk}$, $\text{diag}(\delta_{ij}) = (1, 1, 1)$. Under these assumptions it follows that

$$U = e^{X^i J_i} = I \cos \|X\| + 2\hat{X}^i J_i \sin \|X\|, \quad (2)$$

is an element of $SU(2)$ where $\|X\| = \sqrt{X^i X^j \delta_{ij}}$, $\hat{X}^i = \frac{X^i}{\|X\|}$, and I is the 2×2 identity matrix. Using this to compute the $su(2)$ connection 1-form $\bar{A} \equiv \bar{A}^i J_i = U^{-1} dU$, we get

$$\bar{A}^i = \frac{\sin \|X\|}{\|X\|} dX^i + \frac{(\|X\| - \sin \|X\|)}{\|X\|^2} X^i d\|X\| - \frac{2}{\|X\|^2} \sin^2 \left(\frac{\|X\|}{2} \right) \varepsilon^i{}_{jk} X^j dX^k. \quad (3)$$

By construction it follows that the curvature of \bar{A}^i vanishes, $\bar{F}^i = d\bar{A}^i + \frac{1}{2}\varepsilon^i{}_{jk}\bar{A}^j \wedge \bar{A}^k = 0$. Therefore, on-shell (i.e., using the equations of motion $\bar{F}^i = 0$) the Chern-Simons action principle (1) acquires the form

$$S[\bar{A}] = -\frac{\kappa}{4\pi} \frac{1}{3!} \int_{\mathcal{M}^3} \varepsilon_{ijk} \bar{A}^i \wedge \bar{A}^j \wedge \bar{A}^k, \quad (4)$$

which can be rewritten using (3) as

$$\begin{aligned} S &= \frac{\kappa}{2\pi} \frac{1}{3!} \int_{\mathcal{M}^3} \frac{(\cos \|X\| - 1)}{\|X\|^2} \varepsilon_{ijk} dX^i \wedge dX^j \wedge dX^k \\ &= \frac{\kappa}{4\pi} \int_{\mathcal{M}^3} d \left[\frac{(\sin \|X\| - \|X\|)}{\|X\|^3} \varepsilon_{ijk} X^i dX^j \wedge dX^k + dH \right] \\ &= \frac{\kappa}{4\pi} \int_{\partial \mathcal{M}^3} \left[\frac{(\sin \|X\| - \|X\|)}{\|X\|^3} \varepsilon_{ijk} X^i dX^j \wedge dX^k + dH \right], \end{aligned} \quad (5)$$

and because $\partial \mathcal{M}^3$ has no boundary, $\partial(\partial \mathcal{M}^3) = \emptyset$, we finally get

$$S[X^i] = \frac{\kappa}{4\pi} \int_{\partial \mathcal{M}^3} \frac{(\sin \|X\| - \|X\|)}{\|X\|^3} \varepsilon_{ijk} X^i dX^j \wedge dX^k. \quad (6)$$

This is the action principle induced by the Chern-Simons theory when the latter is defined on a three-manifold \mathcal{M}^3 with a boundary $\partial \mathcal{M}^3$. In order to get the second line of Eq. (5) we have assumed that the third cohomology group H^3 of \mathcal{M}^3 vanishes, $H^3(\mathcal{M}^3) = \emptyset$, which guarantees that the three-form in the integrand of the first line of Eq. (5) is globally exact [1].

Before going on, notice that the model (6) belongs to the class of theories defined by the action principle

$$S[X^i] = \frac{\kappa}{4\pi} \int_{\partial \mathcal{M}^3} f(\|X\|) \varepsilon_{ijk} X^i dX^j \wedge dX^k, \quad (7)$$

for the particular choice of f given by

$$f(\|X\|) = \frac{\sin \|X\| - \|X\|}{\|X\|^3}. \quad (8)$$

The model is, essentially, the integral of the pull-back of the symplectic structure $\omega = f(\|X\|) \varepsilon_{ijk} X^i dX^j \wedge dX^k$ to the boundary $\partial \mathcal{M}^3$. The symplectic structure ω

can be thought as living in \mathbb{R}^3 , which is isomorphic to $su(2)$ -the Lie algebra of $SU(2)$ -and whose points are labeled by the coordinates X^i . The symplectic structure ω is degenerate because it is a 3×3 antisymmetric matrix.

In order to count the physical degrees of freedom of the field theory defined by the action principle (6), Dirac's canonical analysis is employed. It is convenient to perform the analysis using (7) instead of (6). We take $\mathcal{M}^3 = \mathbb{R} \times D^2$ where D^2 is a two-dimensional disc. Therefore, $\partial\mathcal{M}^3 = \mathbb{R} \times S^1$ where the circle S^1 is the boundary of D^2 , $S^1 = \partial D^2$. Note that $H^3(\mathbb{R} \times D^2) = \emptyset$ [1]. Let (τ, σ) be local coordinates that label the points of $\mathbb{R} \times S^1$, the time coordinate τ labels the points of \mathbb{R} and the space coordinate σ labels the points of S^1 . Therefore, $dX^i = \partial_\tau X^i d\tau + \partial_\sigma X^i d\sigma \equiv \dot{X}^i d\tau + \partial_\sigma X^i d\sigma$ and the action (7) becomes

$$S[X^i] = \frac{\kappa}{2\pi} \int_{\mathbb{R}} d\tau \int_{S^1} d\sigma f(\|X\|) \varepsilon_{ijk} X^i \dot{X}^j \partial_\sigma X^k. \quad (9)$$

The definition of the momenta p_i , canonically conjugate to X^i , implies that we have three primary constraints

$$\gamma_i := p_i + \frac{\kappa}{2\pi} f(\|X\|) \varepsilon_{ijk} X^j \partial_\sigma X^k \approx 0. \quad (10)$$

A straightforward computation implies that the canonical Hamiltonian vanishes, and so the action principle acquires the form

$$S[X^i, p_i, \lambda^i] = \int_{\mathbb{R}} d\tau \int_{S^1} d\sigma \left(\dot{X}^i p_i - \lambda^i \gamma_i \right). \quad (11)$$

Computing the variation of this action with respect to the independent variables we get the dynamical equations

$$\begin{aligned} \dot{X}^i &= \lambda^i, \\ \dot{p}_i &= \frac{\kappa}{2\pi} f \varepsilon_{ijk} (2\lambda^j \partial_\sigma X^k - X^j \partial_\sigma \lambda^k) + \frac{f'}{\|X\|f} \lambda^j (p_j - \gamma_j) X_i \\ &\quad + \frac{\kappa}{2\pi} (\partial_\sigma f) \varepsilon_{ijk} \lambda^j X^k, \end{aligned} \quad (12)$$

as well as the constraints (10). Here f' stands for the derivative of f with respect to $\|X\|$.

Using Eqs. (12) to compute the evolution of the primary constraints γ_i , we obtain

$$\begin{aligned} \dot{\gamma}_i &= \frac{\kappa}{2\pi} \varepsilon_{ijk} \lambda^j [3f \partial_\sigma X^k + (\partial_\sigma f) X^k] \\ &\quad + \frac{f'}{\|X\|f} [\lambda^j (p_j - \gamma_j) X_i + \lambda^j X_j (p_i - \gamma_i)], \end{aligned} \quad (13)$$

and therefore the consistency conditions for the primary constraints $\dot{\gamma}_i \approx 0$ imply a system of three linear and homogeneous equations for the Lagrange multipliers λ^i , given by

$$\frac{\kappa}{2\pi} \varepsilon_{ijk} \lambda^j (3f \partial_\sigma X^k + X^k \partial_\sigma f) + \frac{f'}{\|X\|f} (p_j X_i - p_i X_j) \lambda^j \approx 0. \quad (14)$$

This system of equations has one non-trivial null vector in the generic case, whose components are given by

$$\begin{aligned} v^i(\tau, \sigma) &= \frac{1}{2} \int_{S^1} d\sigma' \varepsilon^{ijk} \{\gamma_j(\sigma), \gamma_k(\sigma')\}, \\ &= \frac{\kappa}{2\pi} (3f \partial_\sigma X^i + X^i \partial_\sigma f) + \frac{f'}{\|X\|f} \varepsilon^i{}_{jk} X^j p^k, \end{aligned} \quad (15)$$

which gives rise to the first-class constraint

$$\gamma := v^i \gamma_i = \frac{\kappa}{2\pi} (3f + \|X\|f') p_i \partial_\sigma X^i \approx 0. \quad (16)$$

The meaning of the gauge symmetry generated by (16) is clear because the constraint (16) is $(3f + \|X\|f')$ times the diffeomorphism constraint. Thus, (16) reflects the fact that the action for the two-dimensional theory at the boundary (6) is diffeomorphism invariant. As a consequence the rank of the system of equations (14) is two, which means that there are two second-class constraints among the γ_i . Thus, the counting of the physical degrees of freedom is as follows. The extended phase space is parametrized by 3 configuration variables X^i and the corresponding 3 canonical momenta π_i , there are 1 first-class and 2 second-class constraints. Therefore the system has $\frac{1}{2}(2 \times 3 - 2 \times 1 - 2) = 1$ physical degree of freedom per point of S^1 .

In summary, the physics involved in a Chern-Simons theory defined on a three-manifold with boundary is completely different from the physics involved in the Chern-Simons theory defined on a three-manifold without boundary. In the former there are local and physical excitations living at the boundary while in the latter there are simply no local excitations. It is just the presence of the boundary that makes one theory be completely different from the other, i.e., it is the boundary what generates-in the sense explained above-the local dynamics at the boundary itself.

3. Adding boundary conditions on the original connection

Now, the issue of the boundary conditions will be analyzed. As we already mentioned, the variation of the action principle (1) with respect to the connection gives the equations of motion $F = 0$ provided that

$$\int_{\partial \mathcal{M}^3} k_{ij} A^i \wedge \delta A^j, \quad (17)$$

vanishes. On the other hand, when the solution for the connection one-form (3) for the particular case of $SU(2)$ is inserted back into the action, the boundary action principle (6) is obtained. Even though the original equations of motion $F = 0$ are obtained keeping some specific boundary conditions on A^i such that (17) vanishes, this information is not incorporated into the resulting action principle (6). This situation is similar to what happens, for instance, in the first order formulation for three-dimensional gravity. There, the action principle depends functionally on the triad e^I and the Lorentz connection $\omega^I{}_J$ and it is given by

$$S[e^I, \omega^I{}_J] = \int_{\mathcal{M}^3} \varepsilon_{IJK} e^I \wedge R^{JK}[\omega], \quad (18)$$

where $R^I{}_J[\omega] = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J$ is the curvature of the Lorentz connection $\omega^I{}_J$. The variation of the action (18) with respect to the independent fields gives the equations of motion $de^I + \omega^I{}_J \wedge e^J = 0$ and $R^{IJ}[\omega] = 0$ provided that $\int_{\partial \mathcal{M}^3} \varepsilon_{IJK} e^I \wedge \delta \omega^{JK}$ vanishes. When we solve $de^I + \omega^I{}_J \wedge e^J = 0$ for the connection $\omega^I{}_J$ in terms of the triad, namely, $\omega^I{}_J[e]$ and plug it back into the action (18), we recover, essentially, the Einstein-Hilbert action $S[e]$ expressed in terms of the triads. There is no need of adding to $S[e]$ the specific boundary conditions on the fields of the original action principle (18) that led to the original equations of motion.

Let us come back to the Chern-Simons theory. One might be interested in keeping the original boundary conditions on the connection that kill (17) in the resulting field

theory at the boundary (6). Due to the fact the boundary action principle depends on the auxiliary fields X^i , the original conditions on the connection must be rewritten, using (3), as conditions on the auxiliary fields X^i . By doing this, these boundary conditions are added to (6) via Lagrange multipliers or as primary constraints if we are interested in the Hamiltonian formulation of the theory. Let us apply these ideas in what follows:

As an illustration, we will take the boundary conditions $A_\tau^i + \varepsilon A_\sigma^i = 0$ on $\mathbb{R} \times S^1$ where $\varepsilon = \pm 1$, which kill (17). We add these boundary conditions via Lagrange multipliers u^i to the action principle (7), which becomes

$$\begin{aligned} S[X^i, u^i] := & \frac{\kappa}{2\pi} \int_{\mathbb{R}} d\tau \int_{S^1} d\sigma \left[f \varepsilon_{ijk} X^i \dot{X}^j \partial_\sigma X^k - \dot{X}^i u^j \left((3f + \|X\|f') \varepsilon_{ijk} X^k \right. \right. \\ & + (\|X\|^2 f + 1) \delta_{ij} - f X_i X_j \Big) - \varepsilon u^j \partial_\sigma X^i \left((3f + \|X\|f') \varepsilon_{ijk} X^k \right. \\ & \left. \left. + (\|X\|^2 f + 1) \delta_{ij} - f X_i X_j \right) \right], \end{aligned} \quad (19)$$

after making the 1 + 1 decomposition. The definition of the momenta (p_i, π_i) , canonically conjugated to the coordinates (X^i, u^i) , generates now six primary constraints,

$$\begin{aligned} \phi_i := & p_i + \frac{\kappa}{2\pi} f \varepsilon_{ijk} X^j \partial_\sigma X^k + \frac{\kappa}{2\pi} u^j \left((3f + \|X\|f') \varepsilon_{ijk} X^k \right. \\ & \left. + (\|X\|^2 f + 1) \delta_{ij} - f X_i X_j \right) \approx 0, \end{aligned} \quad (20)$$

$$\chi_i := \pi_i \approx 0.$$

A straightforward computation implies that the canonical Hamiltonian is

$$H_0 := -\varepsilon p_i \partial_\sigma X^i, \quad (21)$$

and so the action principle acquires the form

$$S[X^i, u^i, p_i, \pi_i, \lambda^i, \Lambda^i] := \int_{\mathbb{R}} d\tau \int_{S^1} d\sigma \left(\dot{X}^i p_i + \dot{u}^i \pi_i + \varepsilon p_i \partial_\sigma X^i - \lambda^i \phi_i - \Lambda^i \chi_i \right). \quad (22)$$

Its variation with respect to the independent variables yields the dynamical equations

$$\dot{X}^i = \lambda^i - \varepsilon \partial_\sigma X^i,$$

$$\dot{u}^i = \Lambda^i,$$

$$\begin{aligned} \dot{\pi}_i = & -\frac{\kappa}{2\pi} \lambda^j \left((3f + \|X\|f') \varepsilon_{jik} X^k + (\|X\|^2 f + 1) \delta_{ij} - f X_i X_j \right), \\ \dot{p}_i = & \frac{\kappa}{2\pi} f \varepsilon_{ijk} (2\lambda^j \partial_\sigma X^k - X^j \partial_\sigma \lambda^k) + \frac{\kappa}{2\pi} (\partial_\sigma f) \varepsilon_{ijk} \lambda^j X^k - \frac{\kappa}{2\pi} \frac{f'}{\|X\|f} X^i \lambda^j (\phi_j - p_j) \\ & - \varepsilon \partial_\sigma p_i - \frac{\kappa}{2\pi} \lambda^j u^k \frac{\partial}{\partial X^i} \left((3f + \|X\|f') \varepsilon_{jkl} X^l + (\|X\|^2 f + 1) \delta_{jk} - f X_j X_k \right) \\ = & \frac{\kappa}{2\pi} f \varepsilon_{ijk} (2\lambda^j \partial_\sigma X^k - X^j \partial_\sigma \lambda^k) + \frac{\kappa}{2\pi} (\partial_\sigma f) \varepsilon_{ijk} \lambda^j X^k - \frac{\kappa}{2\pi} \frac{f'}{\|X\|f} X^i \lambda^j (\phi_j - p_j) \\ & - \varepsilon \partial_\sigma p_i - \frac{\kappa}{2\pi} \lambda^j u^k \left(\left(\frac{4f' + \|X\|f''}{\|X\|} \right) \varepsilon_{jkl} X_i X^l + (2f + \|X\|f') X_i \delta_{jk} \right. \\ & \left. + (3f + \|X\|f') \varepsilon_{ijk} - \frac{f'}{\|X\|} X_i X_j X_k - f(\delta_{ij} X_k + X_j \delta_{ik}) \right), \end{aligned} \quad (23)$$

as well as the constraints (20). By using the equations of motion (23), the primary constraints are evolved in time, which yields

$$\begin{aligned}
\dot{\phi}_i &= \frac{\kappa}{2\pi} \varepsilon_{ijk} \lambda^j (3f \partial_\sigma X^k + (\partial_\sigma f) X^k) + \frac{f'}{\|X\|f} \lambda^j (X_i(p_j - \phi_j) - X_j(p_i - \phi_i)) \\
&\quad + \frac{\kappa}{2\pi} \lambda^j u^k \left(\left(\frac{(\|X\|f' - f)f' - \|X\|ff''}{\|X\|f} \right) (\varepsilon_{jkl} X_i - \varepsilon_{ikl} X_j) X^l \right. \\
&\quad \left. - 2(3f + \|X\|f') \varepsilon_{ijk} X^k - (\|X\|f' + (3 + \|X\|^2)f + 1)(X_i \delta_{jk} - X_j \delta_{ik}) \right) \\
&\quad + \frac{\kappa}{2\pi} \Lambda^j \left((3f + \|X\|f') \varepsilon_{ijk} X^k + (\|X\|^2 f + 1) \delta_{ij} - f X_i X_j \right) - \varepsilon \frac{\kappa}{2\pi} \partial_\sigma (f \varepsilon_{ijk} X^j \partial_\sigma X^k) \\
&\quad - \varepsilon \partial_\sigma p_i - \varepsilon \frac{\kappa}{2\pi} u^j \partial_\sigma \left((3f + \|X\|f') \varepsilon_{ijk} X^k + (\|X\|^2 f + 1) \delta_{ij} - f X_i X_j \right), \\
\dot{\chi}_i &= -\frac{\kappa}{2\pi} \lambda^j \left((3f + \|X\|f') \varepsilon_{jik} X^k + (\|X\|^2 f + 1) \delta_{ji} - f X_j X_i \right), \tag{24}
\end{aligned}$$

and therefore by the consistency condition for primary constraints, $\dot{\phi} \approx 0$ and $\dot{\chi} \approx 0$, we get a system of six inhomogeneous linear equations for the Lagrange multipliers. A straightforward computation shows that the determinant of the corresponding matrix is equal to $\left(\frac{\kappa}{2\pi}\right)^6 [(1 + \|X\|^2 f)^2 + \|X\|^2 (3f + \|X\|f')^2]^2$, which does not vanish generically. Therefore, the six Lagrange multipliers can be fixed: $\lambda^i = 0$ and Λ^i gets a cumbersome expression, which is not displayed here but that involves the inhomogeneous part of the system (24). From this, we conclude that there are no more constraints and that (20) are all second-class. Finally, the counting of local degrees of freedom is as follows. Due to the fact that there are six coordinates (X^i, u^i) and their respective six momenta (p_i, π_i) , then the system has $\frac{1}{2} [2 \times (3 + 3) - 6] = 3$ physical degrees of freedom per point on S^1 .

In summary, the example illustrates the role of boundary conditions on the original boundary field theory (7), namely, the specific choice of the boundary conditions $A_\tau + \varepsilon A_\sigma = 0$ increases the number of local degrees of freedom from one to three. Moreover, these particular boundary conditions kill the gauge freedom of the original boundary field theory. Therefore, we conclude that depending on the specific form for the boundary conditions that we might impose on the original connections, the number of physical degrees could be modified as well as the gauge content of the original boundary theory.

4. Implications: a method to construct field theories at the boundary

One of the main lessons learned from the Chern-Simons theory defined on a three-manifold with boundary is that the results of the previous section provide a method to construct field theories living at the boundary. The method can be summarized in the following steps:

- (i) Start with an action principle for a given theory defined on a n -dimensional manifold \mathcal{M}^n with boundary $\partial\mathcal{M}^n$.
- (ii) Solve the equations of motion and write the Lagrangian \mathbf{L} as $\mathbf{L} = d\mathbf{L}_{n-1}$. It is essential that the n th cohomology group vanishes in order for $\mathbf{L} = d\mathbf{L}_{n-1}$ be globally defined on \mathcal{M}^n .

- (iii) Use Stokes theorem to build a theory defined on $\partial\mathcal{M}^{n-1}$ and given by $S = \int_{\partial\mathcal{M}^{n-1}} \mathbf{L}_{n-1}$.
- (iv) Apply Dirac's canonical analysis to $S = \int_{\partial\mathcal{M}^{n-1}} \mathbf{L}_{n-1}$ in order to unveil its physical local degrees of freedom.

Whether the steps from (i) to (ii) can be carried out or not will depend on the specific Lagrangian \mathbf{L} under consideration, of course. We do not have, at this moment, a criterium that allows us to say which Lagrangians \mathbf{L} will fulfil the properties (i) and (ii). However, if these steps could be implemented in a given theory, this will allow us to study the direct relation between the dynamics on the bulk and its correspondent dynamics on the boundary in an unambiguous way.

5. Concluding Remarks

In this paper we have studied the boundary field theory induced by the $SU(2)$ Chern-Simons theory when this is defined on a three-dimensional manifold with boundary. The canonical analysis applied to the resulting action principle for the theory at the boundary shows that it has one physical local degree of freedom, whose origin is attached to the fact that the three-dimensional manifold has a boundary; there are no (physical) local degrees of freedom when the three-manifold has no boundary. Furthermore, the number of local degrees of freedom as well as the gauge content of the theory can be changed by adding conditions on the gauge potentials of the initial Chern-Simons theory to the action principle for the boundary theory, which has been illustrated for a particular choice of the boundary conditions. This last point is, certainly, not surprising but makes clear the role of boundary conditions on the boundary field theory and the origin of the boundary degrees of freedom of the Chern-Simons theory. Moreover, the techniques used here can be applied to other Lie groups, besides $SU(2)$, and the systematic procedure extracted from the model and described in section 4 can, in principle, be applied to other theories defined in arbitrary spacetimes with boundaries.

For instance, these ideas can be applied to Euclidean gravity in $(2+1)$ dimensions with a negative cosmological constant Λ . The action principle for this theory in the first order formalism

$$S_\Lambda = \alpha \int_{\mathcal{M}^3} \left[e_I \wedge \left(d\omega^I + \frac{1}{2} \varepsilon^I{}_{JK} \omega^J \wedge \omega^K \right) - \frac{\Lambda}{6} \varepsilon_{IJK} e^I \wedge e^J \wedge e^K \right], \quad (25)$$

where α is a constant to adjust the units of the fields, can be expressed as two “independent” Chern-Simons theories [3, 14, 15, 16] by first defining two connection one-forms A_\pm^I in terms of the triad e^I and the Lorentz connection ω^I as follows

$$A_\pm^I = \omega^I \pm \frac{1}{\ell} e^I, \quad (26)$$

with $\Lambda = -1/\ell^2$ and then plugging back e^I and ω^I (in terms of A_\pm^I) into the action principle (25) to rewrite it as

$$\begin{aligned} S_{G\Lambda}[A_+, A_-] = & \frac{\kappa}{4\pi} \int_{\mathcal{M}^3} \left[A_+^I \wedge F_+^J - \frac{1}{3!} f^I{}_{KL} A_+^K \wedge A_+^L \wedge A_+^J \right] k_{IJ} \\ & - \frac{\kappa}{4\pi} \int_{\mathcal{M}^3} \left[A_-^I \wedge F_-^J - \frac{1}{3!} f^I{}_{KL} A_-^K \wedge A_-^L \wedge A_-^J \right] k_{IJ} \\ & - \frac{\kappa}{4\pi} \int_{\partial\mathcal{M}^3} A_+^I \wedge A_-^J k_{IJ}, \end{aligned} \quad (27)$$

$\kappa = \alpha \ell \pi$. The boundary term can not be neglected because it contributes to the boundary dynamics. For the Euclidean signature the gauge group of interest is $SO(3)$, therefore, the Killing-Cartan metric can be normalized to be $k_{IJ} = \delta_{IJ}$. Moreover, the connections A_+ and A_- as functions of the auxiliary fields, X^I and Y^I respectively, have the same form given in (3). Therefore, the boundary field theory induced by (27) is essentially two theories like (7) plus one interaction term. This system is currently being analyzed in [17].

To conclude, we would like to mention some advantages and disadvantages of our method by confronting it with some previous works. The procedure described in section 4 has the following advantages: 1) it is covariant because we use the solutions of the equations of motion in contrast to what happens in Refs. [4, 6], where the solution of the constraints is used, which breaks down covariance. Therefore, in the our approach, the implementation of other interactions is straightforward while in the later is not. 2) The resulting boundary action principle is composed entirely by a surface term from the very beginning in opposition to the resulting action principle of [4, 6] that is composed by one surface term and one bulk term, and additional handling is needed to write the bulk term as a surface term in order to have a boundary field theory. 3) The implementation of the boundary conditions is systematic because we add them via lagrange multipliers or as primary constraints in the Hamiltonian formalism. Thus, Dirac method can be systematic applied to analyze the constraint structure of the theory. In opposition, in [4, 6] it is not clear how to incorporate the boundary conditions. 4) we do not have a criterium to know *a priori* to which type of Lagrangians the method described in section 4 can be applied, besides the Chern-Simons theory. In this sense this a disadvantage of our approach.

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